

Equivariant Todd Classes for Toric Varieties

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Abstract

For a complete toric variety, we obtain an explicit formula for the localized equivariant Todd class in terms of the combinatorial data – the fan. This is based on the equivariant Riemann-Roch theorem and the computation of the equivariant cohomology and equivariant homology of toric varieties.

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1 Introduction

A toric variety is a normal variety with an action of an algebraic torus, which contains the torus as an open dense subset. They form a very interesting family of complex algebraic varieties, and were introduced by Demazure [8] and Kempf-Knudsen-Mumford-Saint-Donat [15]. Every toric variety can be described by a set of combinatorial data - a fan. The theory of toric varieties established a now classical connection between algebraic geometry and the theory of convex polytopes. In particular, the Todd classes of complete toric varieties have important consequences for numerical results on convex polytopes, as observed by Danilov. Since the seventies, several authors have established formulas for Todd classes of toric varieties [16], [17], [13] [6], [7].

Brion and Vergne [5] proved an equivariant version of Riemann-Roch theorem for complete simplicial toric varieties, and in particular they introduced the equivariant Todd classes for simplicial toric varieties with values in equivariant cohomology. We follow this direction to work on the Todd classes of general complete toric varieties. But we use equivariant homology instead of equivariant cohomology, because in the general the Todd classes take values in homology [2].

Basically the paper is divided into two parts. In the first part, we set up the equivariant Riemann-Roch theorem following the framework of Baum-Fulton-MacPherson [2]. For our purpose, we only consider G -varieties (maybe singular) over \mathbb{C} , where G is a complex linear reductive algebraic group. Roughly speaking, the problem that we need to solve is to give correct definitions. To get the proper definition, the key is Totaro's approximation of EG [10]. Based on it, we can define equivariant homology [9] and establish the equivariant Riemann-Roch theorem (Theorem 5.1). Edidin and Graham have established an version of equivariant Riemann-Roch theorem [11], because our result is slightly different from theirs, as we use equivariant homology instead of the equivariant Chow group, for completeness, we include this discussion here.

The second part is the computation. We compute the equivariant cohomology and equivariant homology for toric varieties. Based on that, we obtain a concrete formula for the equivariant Todd class (Theorem 9.4). Here the power maps of toric varieties play a very important role.

For simplicity, in the paper, we assume all cohomology or homology is taken over the rational number field \mathbb{Q} .

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ant Todd classes.

2 Totaro's approximation of EG

The purpose of this section is to describe a construction [10], due to Totaro, of an algebro-geometric substitute for the classifying space of topological group.

For a topological group G , the spaces EG and BG are infinite dimensional, and are hard to control in general. What Totaro produces for a reductive algebraic group is a directed system of algebraic G -bundles $E_n \rightarrow B_n$ with the following property: for any principal algebraic G -bundle $E \rightarrow X$, there is a map $X' \rightarrow X$ with the fiber isomorphic to \mathbb{A}^m , such that the pullback bundle $E' \rightarrow X'$ is pulled back from one of the bundles in the directed system by a map $X' \rightarrow B_n$.

For a complex linear reductive algebraic group G (In fact it works for more generality), we can construct the directed system as follows:

Every object in the directed system is a pair (V, V') , such that V is a representation of G , V' is a non-empty open set (in étale topology) of V with free G action, $V' \rightarrow V'/G$ is a principal G -bundle. $(V, V') < (W, W')$ if V is a subspace of W , $V' \subset W'$, $\text{codim}_V(V - V') < \text{codim}_W(W - W')$. The morphisms are just inclusions.

For some groups, there is a convenient choice of the directed system. For example, if $G = T$ is a complex torus of rank n , then we can take $\{(V^n, (V^n - \{0\})) \mid \dim V = l\}_l$ as a directed system, where V^n has a $T \cong (\mathbb{C}^*)^n$ action: $(t_1, t_2, \dots, t_n)(v_1, v_2, \dots, v_n) = (t_1 v_1, t_2 v_2, \dots, t_n v_n)$.

3 Equivariant cohomology

Let G be a topological group, X be a G -space, Borel [1] defined the equivariant cohomology of X as

$$H_G^*(X) = H^*(X \times^G EG)$$

where $EG \rightarrow BG$ is the universal G -bundle.

There is an interesting property of equivariant cohomology with respect to the change of groups.

PROPOSITION 3.1 *Let H be a closed subgroup of G , X be a H -space, then*

$$H_G^*(G \times^H X) \cong H_H^*(X)$$

Using the directed system in Section 2, we have another description of equivariant cohomology:

LEMMA 3.2 *For a complex Lie group G (which has a directed system as in the previous section) and X a complex algebraic variety on which G operates algebraically,*

$$H_G^*(X) = \varprojlim_{(V, V')} H^*(X \times^G V')$$

where the inverse limit is taken over any directed system satisfying the properties in previous section.

Proof. For any (V, V') in the directed system, $V' \rightarrow V'/G$ is a principal G -bundle, so we have a commutative diagram:

$$\begin{array}{ccc} V' & \rightarrow & EG \\ \downarrow & & \downarrow \\ V'/G & \rightarrow & BG \end{array}$$

There is an induced map: $X \times^G V' \rightarrow X \times^G EG$ and we obtain a map: $H_G^*(X) \rightarrow H^*(X \times^G V')$, thus a map $H_G^*(X) \rightarrow \varprojlim_{(V, V')} H^*(X \times^G V')$.

By the properties of the directed system, we can prove this map is an isomorphism. \square

For every G -vector bundle E over X , we can check $E \times^G V' \rightarrow X \times^G V'$ and $E \times^G EG \rightarrow X \times^G EG$ are all G -vector bundles, so we can define the Chern character $ch^G(E) \in H_G^*(X)$ and $ch^V(E) \in H^*(X \times^G V')$ by

$$ch^G(E) = ch(E \times^G EG)$$

and

$$ch^V(E) = ch(E \times^G V')$$

We see that under the map $H_G^*(X) \rightarrow H^*(X \times^G V')$, $ch^G(E)$ goes to $ch^V(E)$.

In the same way, we can define the cohomological equivariant Todd class for a G -vector bundle over a smooth X with smooth G -action. For any (V, V') in the directed system, $X \times^G V'$ is smooth. So we can define the cohomological equivariant Todd class $Td^G(X)$ to be the element in $H_G^*(X)$ which maps to $Td(X \times^G V')$ for every (V, V') in the directed system.

4 Equivariant homology

Now for a complex linear reductive algebraic group G , we can define the equivariant homology [9] for a G -variety X as

$$H_i^G(X) = H_{i+2l-2g}^{BM}(X \times^G V'),$$

where $\{(V, V')\}$ is a directed system for G and $\text{codim}_V(V - V')$ is big enough, H_*^{BM} is Borel-Moore homology, $l = \dim_{\mathbb{C}} V$, $g = \dim_{\mathbb{C}} G$. It is possible that $H_i^G(X) \neq 0$ for negative i .

PROPOSITION 4.1 $H_i^G(X)$ is independent of (V, V') if $\text{codim}_V(V - V')$ is big enough.

Proof. See [9]. \square

Just as for equivariant cohomology, we have the following property.

PROPOSITION 4.2 Let H be a closed subgroup of G , X be a H -space, then

$$H_*^G(G \times^H X) \cong H_*^H(X)$$

As usual, $H_*^G(X)$ is a module over $H_G^*(X)$. In other words there is a cap product: $\cap : H_G^*(X) \otimes H_*^G(X) \rightarrow H_*^G(X)$ defined in the following way:

If $\alpha \in H_G^*(X)$, then for any (V, V') , the image of α under the map $H_G^*(X) \rightarrow H^*(X \times^G V')$ defines a map by cap product: $H_*^{BM}(X \times^G V') \rightarrow H_*^{BM}(X \times^G V') \rightarrow H_*^G(X)$, thus all these induce a map from $H_*^G(X)$ to $H_*^G(X)$. This gives us the module structure.

Now let us define the orientation class $[X]_G \in H_{2n}^G(X)$ for a variety X of complex dimension n .

For any object (V, V') in the directed system, $X \times^G V'$ defines a class $[X \times^G V'] \in H_{2n+2l-2g}^{BM}(X \times^G V')$, when $\text{codim}_V(V - V')$ is big enough, this is our orientation class $[X]_G$.

5 Equivariant Riemann-Roch Theorem

Now we can define the equivariant Todd class τ^G as follows [11], For any equivariant coherent G -sheaf \mathcal{F} over X (X is a complex algebraic variety equipped with an algebraic G -action), if $p : X \times V' \rightarrow X$ is the projection, then the pullback sheaf

$p^* \mathcal{F}$ is a G -equivariant coherent sheaf over $X \times V'$, it descends to a coherent sheaf \mathcal{F}_V over $X \times^G V'$. We have the Todd class $\tau(\mathcal{F}_V)$ is in $H_*^{BM}(X \times^G V')$. Meanwhile, we have a vector bundle over $X \times^G V' \rightarrow X \times^G (V' \times V)$, its cohomological Todd class $Td(X \times^G (V' \times V))$ is in $H^*(X \times^G V')$ and is invertible. Thus we get a map $K_0^G(X) \rightarrow H_*^{BM}(X \times^G V')$, $\mathcal{F} \mapsto \tau(\mathcal{F}_V) \cap Td^{-1}(X \times^G (V' \times V))$. We can check, it is compactible with the transition maps, so we get a map $K_0^G(X) \rightarrow \Pi_p H_p^G(X) = \hat{H}_*^G(X)$, that is τ^G .

THEOREM 5.1 [*Equivariant Riemann-Roch Theorem*] τ^G is a natural transformation (with respect to G -equivariant proper maps) from $K_0^G(X)$ to $\hat{H}_*^G(X)$, such that for any G -variety X , we have the following commutative diagram:

$$\begin{array}{ccccc} K_0^G(X) & \otimes & K_0^G(X) & \xrightarrow{\otimes} & K_0^G(X) \\ & \downarrow & ch^G \otimes \tau^G & & \downarrow \tau^G \\ \hat{H}_*^G(X) & \otimes & \hat{H}_*^G(X) & \xrightarrow{\cap} & \hat{H}_*^G(X) \end{array}$$

and if $f : X \rightarrow Y$ is a proper G -equivariant algebraic map, then we have following commutative diagram

$$\begin{array}{ccc} K_0^G(X) & \xrightarrow{f_*} & K_0^G(Y) \\ \downarrow \tau^G & & \downarrow \tau^G \\ \hat{H}_*^G(X) & \xrightarrow{f_*} & \hat{H}_*^G(Y) \end{array}$$

Moreover, if X is non-singular, and \mathcal{O}_X is the structure sheaf, then

$$\tau^G(\mathcal{O}_X) = Td^G(X) \cap [X]_G$$

Furthermore if any morphism $\phi : K_0^G(X) \rightarrow \hat{H}_*^G(X)$ satisfies the above properties and for any free G -space X , $\phi(X) = \tau(X/G)$, the normalized Baum-Fulton-MacPherson Todd class of X/G , then $\phi = \tau^G$.

Proof. We can directly check the properties of τ^G by definition. As the uniqueness, it follows from the fact that any space V' is a free G -space, so is $X \times V'$, thus ϕ agrees with τ^G on $X \times^G V'$, passing to the limit, we know $\phi = \tau^G$. \square

6 Equivariant cohomology of toric varieties

Having defined the equivariant Todd class, now let us turn to toric varieties and describe the equivariant Todd classes concretely. Before further discussion, let us recall some general facts about toric varieties.

Let $T \cong (\mathbb{C}^*)^d$ be an algebraic torus of dimension d , $N \cong \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^d$ be the set of 1-parameter subgroups of T , Σ be a fan, (ie, a family of strongly convex rational cones in $N \otimes \mathbb{Q}$ such that the intersection of any two elements is again an element in the family and each face of a cone of the family is in the family), X be the toric variety associated to it. If we denote the dual lattice of N by M , then M is isomorphic to the character group of the torus T .

For any $\sigma \in \Sigma$, let us introduce some notation related to it:

- U_σ : –the affine toric variety corresponding to it
- O_σ : –the corresponding T – orbit (which is closed in U_σ)
- σ^\perp : $= \{u \in M_\mathbb{Q} \mid u|_\sigma = 0\}$
- $\check{\sigma}$: $= \{u \in M_\mathbb{Q} \mid \langle u, v \rangle \geq 0, v \in \sigma\}$
- $\bar{\sigma}$: $= \check{\sigma} \cap M$
- S_σ : –the symmetric algebra over $M/\sigma^\perp \cap M$ over \mathbb{Q}
- T_σ : –the subgroup of T with character group $M/\sigma^\perp \cap M$

Notice that the group ring $\mathbb{C}[\bar{\sigma}]$ is the regular function ring of U_σ and $U_\sigma \rightarrow O_\sigma$ is a T -equivariant deformation retraction. In fact $U_\sigma \cong T \times^{T_\sigma} F_\sigma$, where F_σ is an affine T_σ -toric variety, more explicitly, σ spans a linear subspace N_σ of $N \otimes \mathbb{Q}$, in which we have a natural fan consists of σ and its faces, F_σ is the corresponding toric variety.

Let us use Σ' to denote the subset of maximal cones in Σ . Σ has a partition $\{\Sigma(i)\}$ according to the dimension of cones, $\Sigma(i) = \{\sigma \in \Sigma \mid \dim \sigma = i\}$.

For a T -space Y , following Borel [1], $H_T^*(Y)$ by definition is $H^*(Y \times^T ET)$, where $ET \rightarrow BT$ is the universal T -bundle. When Y is a point, we have $H_T^*(\text{point}) = H^*(BT) \cong \mathbb{Q}[x_1, x_2, \dots, x_d]$.

LEMMA 6.1

$$H_T^*(U_\sigma) = S_\sigma$$

Proof. We already know that, as a T -space, U_σ is T -homotopic to $O_\sigma \cong T/T_\sigma$, so $H_T^*(U_\sigma) \cong H_T^*(T/T_\sigma) \cong H_{T_\sigma}^*(\text{point}) \cong S_\sigma$. \square

LEMMA 6.2 *If $\tau \subset \sigma$ is a face, we have the inclusion $i : U_\tau \rightarrow U_\sigma$ [12], then $i^* : H_T^*(U_\sigma) \rightarrow H_T^*(U_\tau)$ is the map $S_\sigma \rightarrow S_\tau$ induced by $M \otimes \mathbb{Q}/\sigma^\perp \rightarrow M \otimes \mathbb{Q}/\tau^\perp$.*

Proof. See [5]. \square

For the toric variety X associated with the given fan Σ , notice that $X \times^T ET$ has an open covering $\mathcal{C} : \{U_\sigma \times^T ET\}_{\sigma \in \Sigma'}$, thus we obtain a spectral sequence which converges to $H_T^*(X)$ [14] with

$$E_1^{p,q} = \bigoplus_{\{\sigma_0, \sigma_1, \dots, \sigma_p\} \subset \Sigma'} H^q(U_{\sigma_0 \cap \sigma_1 \cap \dots \cap \sigma_p} \times^T ET)$$

THEOREM 6.3 *This spectral sequence degenerates at E_2 , and we have a canonical isomorphism of graded $H^*(BT)$ algebras,*

$$Tot(E_2) \cong H_T^*(X)$$

We will prove this theorem in next section. Therefore, every element in $H_T^n(X)$ can be represented as a class $(a_{\sigma_0, \sigma_1, \dots, \sigma_p, q})$ with $a_{\sigma_0, \sigma_1, \dots, \sigma_p, q} \in H^q(U_{\sigma_0 \cap \sigma_1 \cap \dots \cap \sigma_p} \times^T ET)$ and $p + q = n$.

In case X is simplicial, notice that for any q , the sequence

$$0 \rightarrow E_1^{0,q} \rightarrow E_1^{1,q} \rightarrow E_1^{2,q} \rightarrow \dots$$

is exact except at $E_1^{0,q}$, we see

THEOREM 6.4 [5] *If X is simplicial, then $H_T^*(X)$ is isomorphic to the algebra of continuous piecewise polynomial functions.*

7 The power maps of toric varieties

This is a very special property of toric varieties. For any positive integer n , we have a map:

$$\begin{aligned} l_n : \Sigma &\rightarrow \Sigma \\ x &\mapsto nx \end{aligned}$$

It induces a map [12]: $X \rightarrow X$, which we again denote by l_n . If we restrict l_n to the open dense orbit T , then it is the map $T \rightarrow T : t \mapsto t^n$, so we call l_n the n -th power map. l_n is not T -equivariant in general (except for $n = 1$). Let us first consider some properties of these power maps.

LEMMA 7.1 *For a continuous group homomorphism $\phi : G_1 \rightarrow G_2$ of topological groups, ϕ induces a G_1 -equivariant map $EG_1 \rightarrow EG_2$ and a commutative diagram:*

$$\begin{array}{ccc} EG_1 & \xrightarrow{\phi} & EG_2 \\ \downarrow & & \downarrow \\ BG_1 & \xrightarrow{\phi} & BG_2 \end{array}$$

X has a canonical T -action. Based on this action, we can construct a family of T -actions $\{\psi_n\}$ on X :

$$\begin{aligned}\psi_n : T \times X &\rightarrow X \\ (t, x) &\mapsto t^n \cdot x\end{aligned}$$

We will denote by ${}_nX$ the space X , on which T acts by ψ_n . Then $l_n : X \rightarrow {}_nX$ is an equivariant T -map.

Apply the above lemma to the n -th power endomorphism of T : $l_n : T \rightarrow T$ we get a commutative diagram of principal T -fibration:

$$\begin{array}{ccc} ET & \xrightarrow{l_n} & {}_nET \\ \downarrow & & \downarrow \\ BT & \xrightarrow{l_n} & BT \end{array}$$

LEMMA 7.2 $l_n : BT \rightarrow BT$ induces an isomorphism $l_n^* : H^*(BT) \rightarrow H^*(BT)$, and l_n^* acts on $H^{2q}(BT)$ by n^q .

Proof. When $d = 1$, we have a concrete model for BT : \mathbb{CP}^∞ and the map l_n is nothing but $[x_0 : x_1 : x_2 \cdots] \mapsto [x_0^n : x_1^n : x_2^n \cdots]$. For general d , we can take $BT \cong (\mathbb{CP}^\infty)^d$, and in a similar way, we can use the Künneth theorem to prove this lemma. \square

LEMMA 7.3 For any T -space X , we have

$$H_T^*(X) \cong H^*(X \times^T {}_nET)$$

Proof. We have a Cartesian diagram of fibrations with fiber X :

$$\begin{array}{ccc} X \times^T ET & \xrightarrow{id \otimes l_n} & X \times^T {}_nET \\ \downarrow & & \downarrow \\ BT & \xrightarrow{l_n} & BT \end{array}$$

So we get a map between Leray spectral sequences; because BT is simply connected, the E_2 terms are given by tensor product. This implies that the map between Leray spectral sequences yields an isomorphism at the E_2 level. This proves our claim. \square

LEMMA 7.4 Under the isomorphism $H_T^*(U_\sigma) \cong S_\sigma \cong \mathbb{Q}[x_1, x_2, \dots, x_{\dim \sigma}]$, the induced map $(id \otimes l_n)^* : H_T^*(U_\sigma) \rightarrow H_T^*(U_\sigma \times^T {}_nET)$ sends x_i to nx_i .

Now we can prove Theorem 6.3.

Proof of Theorem 6.3 Consider the map: $id \otimes l_n : X \times^T ET \rightarrow X \times^T {}_n ET$, the open covering \mathcal{C} of $X \times^T ET$ and the covering $\mathcal{C}' = \{U_\sigma \times^T {}_n ET\}$ of $X \times^T {}_n ET$. The map $id \otimes l_n$ is compatible with the coverings \mathcal{C} and \mathcal{C}' , so we get a map between the spectral sequences for $X \times^T ET$ and $X \times^T {}_n ET$, which we constructed in previous section.

So we have the diagram:

$$\begin{array}{ccc} E_r^{p,q} & \xrightarrow{d_r} & E_r^{p+r, q-r+1} \\ \downarrow & & \downarrow \\ E_r^{p,q} & \xrightarrow{d_r} & E_r^{p+r, q-r+1} \end{array}$$

where the downarrow maps are the map $(id \otimes l_n)^*$. If we identify $E_r^{p,q}$ as a quotient of a subset of $E_1^{p,q}$, the map $(id \otimes l_n)^*$ is multiplication by $n^{[q/2]}$. So we see, $d_r = 0$ for $r > 1$.

Because this spectral sequence degenerates at E_2 , for any p, q , we have a filtration of $H_T^{p+q}(X)$:

$$H_T^{p+q}(X) = F^0 H_T^{p+q}(X) \supset F^1 H_T^{p+q}(X) \cdots \supset F^k H_T^{p+q}(X) \cdots$$

and the following exact sequence:

$$0 \rightarrow F^{p+1} H_T^{p+q}(X) \rightarrow F^p H_T^{p+q}(X) \rightarrow E_2^{p,q} \rightarrow 0$$

Now using the maps $(id \otimes l_n)_*$, we obtain the isomorphism of $H^*(BT)$ algebras:

$$Tot(E_2) \cong H_T^*(X)$$

□

8 Equivariant homology for toric varieties

The equivariant homology is defined as $H_i^T(X) = H_{i+2(k-1)d}^{BM}(X \times^T V_k)$, where $V_k = (\mathbb{C}^k - \{0\})^d$ with the diagonal action of T , $(t_1, t_2, \dots, t_d) \cdot (v_1, v_2, \dots, v_d) = (t_1 v_1, t_2 v_2, \dots, t_d v_d)$. As in section 7, ${}_n V_k$ means V_k with the action of n -th power of T .

LEMMA 8.1 For any $\sigma \in \Sigma$,

$$H_*^T(O_\sigma) \cong S_\sigma[O_\sigma]$$

In this case, for any $a \in M/\sigma^\perp \cap M$, $\deg a = -2$, and $[O_\sigma]$ is the orientation class, $\deg [O_\sigma] = 2\text{codim } \sigma$.

Proof. We know $O_\sigma \cong T/T_\sigma$, so the statement reduces to the case of one point by Proposition 4.2. \square

For any T -space X , we have a map $\text{id} \otimes l_n : X \times^T V_k \rightarrow X \times^T {}_n V_k$, so similarly to the proof of Lemma 7.3, we can prove

LEMMA 8.2 For any T -space X ,

$$\lim_{\vec{k}} H_*^{BM}(X \times^T V_k) \cong \lim_{\vec{k}} H_*^{BM}(X \times^T {}_n V_k)$$

In other words, we can use either $\{V_k\}$ or $\{{}_n V_k\}$ to define the equivariant homology.

Let us denote the map $\lim_{\vec{k}} H_*^{BM}(X \times^T V_k) \cong \lim_{\vec{k}} H_*^{BM}(X \times^T {}_n V_k)$ by $(\text{id} \otimes l_n)_*$

LEMMA 8.3 Under the isomorphism $H_*^T(O_\sigma) \cong S_\sigma[2\text{codim } \sigma]$, the map: $(\text{id} \otimes l_n)_*$ maps $a \in M/\sigma^\perp \cap M$ to na , i.e., it acts on $H_{2k}^T(O_\sigma)$ as multiplication by $n^{\text{codim } \sigma - k}$.

Proof. It is similar to the proof of Lemma 7.2. \square

Let $X_k = \cup_{\dim \sigma \geq d-k} O_\sigma$, then X_k is closed and we obtain a T -invariant filtration of X ,

$$\{X_0 \subset X_1 \subset \cdots \subset X_n = X\}$$

LEMMA 8.4 If $Z \subset X$ is a closed T -subset of X , then we have following long exact sequence:

$$\cdots \rightarrow H_p^T(Z) \rightarrow H_p^T(X) \rightarrow H_p^T(X-Z) \rightarrow H_{p-1}^T(Z) \rightarrow \cdots$$

Proof. This comes from the facts that we have an exact sequence for the pair $(X \times^T V_k, Z \times^T V_k)$ and taking direct limits is an exact functor. \square

LEMMA 8.5 For any p and k , we have $H_{2k+1}^T(X_p) = 0$

Proof. Notice X_0 is a finite set, and for any $0 \leq p \leq d$, $X_p - X_{p-1}$ is the disjoint union of p -dimensional orbits, so the result follows by induction. \square

THEOREM 8.6 *For any k , we have a canonical decomposition*

$$H_{2k}^T(X) \cong \oplus_{\sigma} H_{2k}^T(O_{\sigma})$$

Therefore, every element in $H_n^T(X)$ can be represented as (b_{σ}) with $b_{\sigma} \in H_n^T(O_{\sigma})$.

Proof. By the above lemmas, for any p and k , we have an exact sequence:

$$0 \rightarrow H_{2k}^T(X_{p-1}) \rightarrow H_{2k}^T(X_p) \rightarrow H_{2k}^T(X_p - X_{p-1}) \rightarrow 0$$

Now considering the map $(id \otimes l_n)_*$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \rightarrow & H_{2k}^T(X_{p-1}) & \rightarrow & H_{2k}^T(X_p) & \rightarrow & H_{2k}^T(X_p - X_{p-1}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H_{2k}^T(X_{p-1}) & \rightarrow & H_{2k}^T(X_p) & \rightarrow & H_{2k}^T(X_p - X_{p-1}) & \rightarrow & 0 \end{array}$$

where the downarrow maps are the maps $(id \otimes l_n)_*$. If $i: X_p - X_{p-1} \rightarrow X_p$ is the inclusion, for any $a \in H_{2k}^T(X_p)$, then $i_*(id \otimes l_n)_*(a) = n^{p-k}i_*(a)$, so $(id \otimes l_n)_*(a) = n^{p-k}a + b$, where $b \in H_{2k}^T(X_{p-1})$. It is easy to see there exists $c \in H_{2k}^T(X_{p-1})$ such that $(id \otimes l_n)_*(c) = n^{p-k}c + b$, that implies $(id \otimes l_n)_*(a + c) = n^{p-k}(a + c)$. Now it is easy to see that $\text{Ker}((id \otimes l_n)_* - n^{p-k}) \cong H_{2k}^T(X_p, X_{p-1})$ and the short exact sequence splits. So by induction, we know $H_{2k}^T(X) \cong \oplus_{\sigma} H_{2k}^T(O_{\sigma})$ and the map $(id \otimes l_n)_*$ is scalar multiplication by $n^{\text{codim } \sigma - k}$ on the summand $H_{2k}^T(O_{\sigma})$. \square

So far, we explored the structure of $H_*^T(X)$ as an Abelian group. As usual, $H_*^T(X)$ is a $H_T^*(X)$ thus a $H^*(BT)$ -module, so let us consider the module structure now.

LEMMA 8.7 *The homology class $[\bar{O}_{\sigma}] \in H_*^T(X)$ under the canonical isomorphism is (a_{τ}) , where $a_{\tau} = 1$ if $\tau = \sigma$, $= 0$ otherwise.*

Proof. Let $p = \text{codim } \sigma$, then $[\bar{O}_{\sigma}] \in H_{2p}^T(X_p) \subset H_{2p}^T(X)$ is of degree $2p$, and the exact sequence we used in Theorem 8.6 is

$$0 \rightarrow H_{2p}^T(X_p) \rightarrow H_{2p}^T(X_p - X_{p-1}) \rightarrow 0$$

so the lemma is clear. \square

$H_*^T(X)$ is generated as an $H^*(BT)$ -module by the classes $\{[\bar{O}_{\sigma}]\}$, we have relations of the following type.

LEMMA 8.8 For any $\sigma \in \Sigma$, $l \in \sigma^\perp$ we have

$$l \cap [\bar{O}_\sigma] = \sum_{\tau} \langle l, n_{\sigma\tau} \rangle [\bar{O}_\tau]$$

where the summation is taken over all these τ 's containing σ as a facet, $n_{\sigma\tau} \in N/(\sigma \cap N)$ is the unique generator of the semigroup $(\tau \cap N)/(\sigma \cap N)$.

So, combining this lemma with Theorem 8.6, we see that $H_*^T(X)$ is the $H^*(BT)$ -module, generated by $\{[\bar{O}_\sigma] \mid \sigma \in \Sigma\}$, modulo the relations given in Lemma 8.8. It is isomorphic to the equivariant Chow group [4].

An immediate consequence of this Lemma 8.8 is the localization theorem for equivariant homology.

COROLLARY 8.9 (*Localization theorem*) For a toric variety X , $i : X^T \rightarrow X$ induces an isomorphism of equivariant homologies up to $H^*(BT)$ torsion.

In the case of complete simplicial fans, this description of equivariant homology is dual to the equivariant cohomology. Assuming X is complete and simplicial, Brion-Vergne proved in [5] that $H_T^*(X)$ is isomorphic to the algebra R_Σ of continuous piecewise polynomial functions on Σ . We will write down the Poincare duality isomorphism PD between R_Σ and $H_*^T(X)$ explicitly.

LEMMA 8.10 For a complete simplicial toric variety X of complex dimension d associated to the fan Σ , we have an $H^*(BT)$ -module morphism $PD : R_\Sigma^* \cong H_*^T(X)$, such that for any $k \in \mathbb{Z}$

$$PD : R_\Sigma^k \cong H_{2d-2k}^T(X)$$

where R_Σ^k is the space of continuous piecewise polynomial functions of homogeneous degree k .

Proof. For $\sigma \in \Sigma$, the subgroup of N generated by the intersections of N and all edges of σ is a subgroup of finite index in the subgroup of N generated by $N \cap \sigma$, let us denote the index by $\text{mult}(\sigma)$.

For any $\tau \in \Sigma(1)$, there is a unique piecewise polynomial function ξ_τ on Σ , the so called Courant function associated to the edge τ [3]. In fact, ξ_τ is a piecewise linear function, which can be described as follows. If σ is a maximal cone, spanned by τ, v_2, \dots, v_d , and if $\{\tau^*, v_2^*, \dots, v_d^*\}$ is the dual basis of $M \otimes \mathbb{Q}$, then $\xi_\tau|_\sigma = \tau^*$. It is clear that $\xi_\tau|_\sigma = 0$ if σ does not contain τ .

Thus for $\sigma \in \Sigma$, we have a continuous piecewise polynomial function ϕ_σ on Σ , of degree $2\dim\sigma$:

$$\phi_\sigma = \text{mult}(\sigma) \Pi_{\tau \in \sigma(1)} \xi_\tau$$

ϕ_σ vanishes identically on all cones which do not contain σ .

Now we can define two $H^*(BT)$ -module homomorphisms. By [3], R_Σ is a $S(M)$ -module with generators $\{\phi_\sigma\}$, where $S(M)$ is the symmetric algebra of M over \mathbb{Q} . So we can define a map $F : R_\Sigma \rightarrow H_*^T(X) : \sum a_\sigma(v) \phi_\sigma \mapsto \sum (-1)^{\dim\sigma} a_\sigma(-v) [\bar{O}_\sigma]$. Obviously it is a $H^*(BT)$ -module homomorphism. Also we can define a natural homomorphism: $G : H_*^T(X) \rightarrow R_\Sigma$, which is given by $\sum a_\sigma(v) [\bar{O}_\sigma] \mapsto \sum (-1)^{\dim\sigma} a_\sigma(-v) \phi_\sigma$.

For a simplicial toric variety, as $H^*(BT)$ -modules, $H_*^T(X)$ and $H_*^T(X)$ have no torsion. If we take the multiplicative set $L \subset H^*(BT)$ to be $\{\prod m_i \mid m_i \in M - \{0\}\}$, then both $H_*^T(X) \rightarrow L^{-1}H_*^T(X)$ and $H_*^T(X) \rightarrow L^{-1}H_*^T(X)$ are embeddings.

We can describe the localized maps $L^{-1}F$ and $L^{-1}G$ as follows. Any $f \in R_\Sigma$ is determined by $\{f|_\sigma\}_{\sigma \in \Sigma'}$, where $f|_\sigma$ is the restriction of f to the maximal cone σ . The localized map $L^{-1}F : L^{-1}R_\Sigma \rightarrow L^{-1}H_*^T(X)$ is given by

$$L^{-1}F(f) = \sum_{\sigma \in \Sigma'} \frac{f|_\sigma}{\phi_\sigma} [\bar{O}_\sigma]$$

It is an isomorphism with inverse map $L^{-1}G$.

So from the following commutative diagram:

$$\begin{array}{ccc} R_\Sigma & \xrightarrow{F} & H_*^T(X) \\ \downarrow & & \downarrow \\ L^{-1}R_\Sigma & \xrightarrow{L^{-1}F} & L^{-1}H_*^T(X) \end{array}$$

we see F is a module isomorphism (with inverse G). This is the Poincare duality isomorphism, let us use PD to denote it. \square

Remark From the proof we see, for an element in $L^{-1}H_*^T(X)$, if its image under $L^{-1}PD^{-1}$ is in R_Σ , then it is in fact in the image of $H_*^T(X) \rightarrow L^{-1}H_*^T(X)$, thus it is an element of $H_*^T(X)$. We will use this result in the next section.

For a general fan, the associated toric variety may be singular. In the category of toric varieties, we have an easy way to resolve the singularities - subdividing the fan. The equivariant homologies behave as follows under subdivision:

LEMMA 8.11 *If Σ_1 is a subdivision of Σ , then the induced map of $f : X_{\Sigma_1} \rightarrow X_\Sigma$ is the following: $f_*(\sum a_{\sigma'} [\bar{O}_{\sigma'}]) = \sum b_\sigma [\bar{O}_\sigma]$, with $b_\sigma = \sum a_{\sigma'}$, where we sum over all $\sigma' \in \Sigma_1$ such that: $\sigma' \subset \sigma$ and $\dim \sigma' = \dim \sigma$.*

Proof. Notice that f induces an isomorphism $O_{\sigma'} \cong O_{\sigma}$ and f_* is an $H^*(BT)$ -homomorphism. \square

9 Equivariant Todd classes

Let M be a lattice of rank d , we will use $\mathbb{Q}[M]$ to denote the group ring of M over \mathbb{Q} and $S(M)$ to denote the symmetric algebra of M over \mathbb{Q} . If $M = \oplus \mathbb{Z}e_i$, then $\mathbb{Q}[M] = \{\sum a_m e^m \mid m \in M\}$ and $S(M) \cong \mathbb{Q}[x_1, x_2, \dots, x_d]$, the polynomial ring of d indeterminates. Let $\hat{S}(M)$ be the completion of $S(M)$, ie, $\mathbb{Q}[[x_1, x_2, \dots, x_d]]$. We have a map $\mathbb{Q}[M] \rightarrow \hat{S}(M)$ which is defined by $e^{x_i} \rightarrow \sum \frac{x_i^n}{n!}$ on generators and extending it to an algebra homomorphism.

Let $\mathbb{Q}[[M]]$ be the set of all formal power series $\sum_{m \in M} a_m e^m$ with rational coefficients. Following Brion-Vergne [5], we call $f \in \mathbb{Q}[[M]]$ summable if there exist $P \in \mathbb{Q}[M]$ and a finite sequence $(m_i)_{i \in I}$, such that in $\mathbb{Q}[[M]]$ we have,

$$f \prod (1 - e^{m_i}) = P,$$

in the case we define the sum of f as:

$$S(f) = P \prod (1 - e^{m_i})^{-1}$$

Notice $S(f)$ is an element of the fraction field of $\mathbb{Q}[M]$.

By this summation, we see easily the following easy but critical result:

LEMMA 9.1 *For any $m \in M$, $\sum_{k=-\infty}^{\infty} e^{km}$ is summable and*

$$S\left(\sum_{k=-\infty}^{\infty} e^{km}\right) = 0$$

LEMMA 9.2 *For any $\sigma \in \Sigma$, $m \in M$, denote by $U_{\sigma} \times \mathbb{C}m \rightarrow U_{\sigma}$ the trivial line bundle on U_{σ} on which T acts by character m , then we have*

$$ch^T(U_{\sigma} \times \mathbb{C}m) = e^m$$

The multiplicity of any character m of T in $O_X(U_{\sigma}) \cong \mathbb{C}[\check{\sigma}]$ is either zero or one, let us denote it by $mult_{\sigma}(m)$. Then for

$$A_{\sigma} = \sum_{m \in \check{\sigma}} mult(m) e^m.$$

we know,

LEMMA 9.3 A_σ is summable.

Proof. See [5]. \square

So we can think of $S(A_\sigma)$ as an element of the quotient field of $\mathbb{Q}[M]$.

THEOREM 9.4 For a complete toric variety X , we have

$$L^{-1}\tau^T(X) = \sum_{\sigma \in \Sigma'} S(A_\sigma)[\bar{O}_\sigma]$$

in $L^{-1}\hat{H}_*^T(X)$.

We will prove this theorem using the following lemmas. Before the proof, we explain what the theorem means. By definition, $S(A_\sigma)$ is a rational function in variables e^{m_i} with possible denominators of the form $\prod(1 - e^{m_i})$, so it can be viewed as a Laurent series in m_i , thus as an element of $L^{-1}\hat{H}_*^T(X)$.

LEMMA 9.5 Theorem 9.4 holds for any smooth toric variety X .

Proof. Because X is smooth, $\tau^T(\emptyset_X) = Td^T(TX) \cap [X]_T$.

If σ is spanned by v_1, v_2, \dots, v_p , and $\{v_1, v_2, \dots, v_d\}$ spans a maximal cone containing σ , $\{v_1^*, v_2^*, \dots, v_d^*\}$ is the dual basis, $i : U_\sigma \rightarrow X$ is the inclusion map, $i^*(TX) \cong U_\sigma \times (\oplus_{i=1}^d \mathbb{C}v_i^*)$. So $Td^T(i^*(TX)) = \prod_{i=1}^d \frac{v_i^*}{1 - e^{-v_i^*}}$, whose image in S_σ is $\prod_{i=1}^p \frac{v_i^*}{1 - e^{-v_i^*}}$. We know $i^*(Td^T(TX)) = Td^T(i^*TX)$, so as a piecewise continuous polynomial function, $Td^T(TX)$ is $\prod_{i=1}^p \frac{v_i^*}{1 - e^{-v_i^*}}$ on σ .

On the other hand, in this case $\text{mult}_\sigma(m, \emptyset_X) = 1$ for $m \in \bar{\sigma}$. For a maximal cone σ spanned by $\{v_1, v_2, \dots, v_d\}$, $\bar{\sigma} = \mathbb{Z}_{\geq 0}v_1^* + \dots + \mathbb{Z}_{\geq 0}v_d^*$, $S(A_\sigma) = \prod_{i=1}^d \frac{1}{1 - e^{v_i^*}}$. So $\sum S(A_\sigma)[\bar{O}_\sigma]$ is an element of $L^{-1}\hat{H}_*^T(X)$.

Notice $L^{-1}PD^{-1}(\sum S(A_\sigma)[\bar{O}_\sigma])$ is equal to $Td^T(X) \in L^{-1}R_\Sigma$. So by the comment after Lemma 8.10, this is the equivariant Todd class. \square

Remark We proved more in this Lemma. We proved that for a smooth toric variety (this is also true for simplicial case), $L^{-1}PD^{-1}(\sum S(A_\sigma)[\bar{O}_\sigma])$ belongs to $\hat{H}_*^T(X)$ and $PD(L^{-1}PD^{-1}(\sum S(A_\sigma)[\bar{O}_\sigma]))$ is the equivariant Todd class.

For a cone σ , let us denote $S(\sum_{m \in \bar{\sigma}} e^m)$ by $S(\mathbb{C}[\bar{\sigma}])$, then we have the following lemma.

LEMMA 9.6 If $\{\sigma_1, \sigma_2, \dots, \sigma_p\}$ is a subdivision of σ , with $\dim \sigma_i = \dim \sigma$, then we have $\sum S(\mathbb{C}[\bar{\sigma}_i]) = S(\mathbb{C}[\bar{\sigma}])$, where both sides are viewed as elements of the fraction field $\mathbb{Q}[M/\sigma^\perp \cap M]$.

Proof. First let us prove this lemma in case $p = 2$. If τ is the common facet of σ_1 and σ_2 , then by [12], there exists $u \in \check{\sigma}_1 \cap (-\check{\sigma}_2) \cap M$, such that $\tau = \sigma_1 \cap u^\perp = (-\sigma_2) \cap u^\perp$, and $\bar{\tau} = \bar{\sigma}_2 + \mathbb{Z}_{\geq 0}u$.

In the vector space spanned by σ , for any facet τ of σ , there is a unique (up to scalar multiplication) $u_\tau \in \check{\sigma} \cap M$, such that $\tau = \sigma \cap u^\perp$ and $\check{\sigma}$ is spanned by all these u_τ 's. So we can assume that $\check{\sigma}$ is spanned by u_1, u_2, \dots, u_p , $\check{\sigma}_1$ is spanned by u_1, u_2, \dots, u_s, u and $\check{\sigma}_2$ is spanned by $u_{s+1}, \dots, u_p, -u$ with $u_i \in M$. Then we see that for any $v \in \check{\sigma}_1 \cup \check{\sigma}_2$, $v + ku \in \check{\sigma}_1 \cup \check{\sigma}_2$ for any $k \in \mathbb{Z}$. So $S([\mathbb{C}[\check{\sigma}_1]] + S([\mathbb{C}[\check{\sigma}_i]])) = S(\sum_{m \in \bar{\sigma}_1 \cup \bar{\sigma}_2} e^m)$. Notice that for any character m of T , it appears in $\bar{\sigma}_i$ at most once, and $\bar{\sigma}_1 \cap \bar{\sigma}_2 = \bar{\sigma}$, by the fact $S(\sum e^{km}) = 0$ for any character m of T , we see $\sum S([\mathbb{C}[\check{\sigma}_i]]) = S([\mathbb{C}[\check{\sigma}]])$ in this case.

For the general case, notice that for a subdivision $\{\sigma_1, \sigma_2, \dots, \sigma_p\}$, we can subdivide them further into $\{\tau_1, \tau_2, \dots, \tau_q\}$, such that σ_i is one of the τ_j 's or the union of two τ_j 's, τ_j and τ_{j+1} have only a common facet, and $\{\tau_1, \tau_2, \dots, \tau_q\}$ can be divided into two sets, $\{\tau_1, \tau_2, \dots, \tau_k\}$ and $\{\tau_{k+1}, \dots, \tau_q\}$, with the following properties:

- $\tau_1 \cup \tau_2 \cup \dots \cup \tau_j$ is a convex cone for any $j \leq k$,
- $\tau_{k+1} \cup \dots \cup \tau_l$ is a convex cone for any $l \leq q$.

So by the result about the case of subdivision into two cones, the lemma is proved. \square

We now return to the proof of theorem 9.4 for a general toric variety X . Take a suitable subdivision Σ_1 of Σ such that the corresponding toric variety X' of Σ_1 is smooth [12]. Then the induced map $f : X' \rightarrow X$ is a proper map, and $f_* \mathcal{O}_{X'} = \mathcal{O}_X$, $R^i f_* \mathcal{O}_{X'} = 0$ for $i > 0$, so $\tau^T(\mathcal{O}_X) = f_* \tau^T(\mathcal{O}_{X'})$. By Lemma 8.8, we know $L^{-1} \tau^T(\mathcal{O}_X)$ is given by the formula. This concludes the induction step. \square

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